

A NEW PROOF FOR SMALL CANCELLATION CONDITIONS OF 2-BRIDGE LINK GROUPS

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ABSTRACT. In this paper, we give a simple proof for the small cancellation conditions of the upper presentations of 2-bridge link groups, which holds the key to the proof of the main result of [1]. We also give an alternative proof of the main result of [1] using transfinite induction.

1. INTRODUCTION

In [1], the second author and M. Sakuma gave a complete characterization of those essential simple loops in a 2-bridge sphere of a 2-bridge link which are null-homotopic in the link complement, and by using the result, they described all upper-meridian-pair-preserving epimorphisms between 2-bridge link groups. The main purpose of this paper is to give a simple proof for the small cancellation conditions of the upper presentations of 2-bridge link groups, which holds the key to the proof of the main result of [1]. We also give an alternative proof of the main result of [1] using transfinite induction. It is well-known that 2-bridge links, $K(r)$, are parametrized by extended rational numbers, r , and that by Shubert's classification of 2-bridge links [5], it suffices to consider $K(r)$ for $r = \infty$ or $0 < r \leq 1$. Here if $r = \infty$ or $r = 1$, then $K(r)$ becomes a trivial 2-component link or a trivial knot, respectively. Since these trivial cases are easy to treat for our purpose (see [1, Section 7]), we may assume $0 < r < 1$. Then such a rational number r is uniquely expressed in the following continued fraction expansion:

$$r = \frac{1}{m_1 + \frac{1}{m_2 + \cdots + \frac{1}{m_k}}} =: [m_1, m_2, \dots, m_k],$$

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where $k \geq 1$, $(m_1, \dots, m_k) \in (\mathbb{Z}_+)^k$, and $m_k \geq 2$.

In [1], the proofs of key lemmas and propositions such as Lemma 7.3 and Propositions 4.3, 4.4 and 4.5 proceed by induction on k , the length of the continued fraction expansion of r , where a rational number \tilde{r} defined by $\tilde{r} = [m_2 - 1, \dots, m_k]$ if $m_2 \geq 2$ and $\tilde{r} = [m_3, \dots, m_k]$ if $m_2 = 1$ plays an important role as a predecessor of $r = [m_1, m_2, \dots, m_k]$ (see [1, Proposition 4.4]).

However, in this paper, we define a well-ordering \preceq on the set of rational numbers greater than 0 and less than 1 (see Definition 4.3), and then prove key lemmas and propositions such as Lemmas 4.4 and 5.1, and Propositions 4.2 and 4.5 using transfinite induction with respect to \preceq , where a rational number \tilde{r} defined by $\tilde{r} = [m_1 - 1, \dots, m_k]$ if $m_1 \geq 2$ or $\tilde{r} = [m_2 + 1, \dots, m_k]$ if $m_1 = 1$ plays a role as a predecessor of $r = [m_1, m_2, \dots, m_k]$ (see Lemma 4.1). Note that having a smaller gap between r and \tilde{r} than in [1] makes the proof less complicated.

This paper is organized as follows. In Section 2, we describe the main statement that we are going to re-prove in the present paper. In Section 3, we recall the upper presentation of a 2-bridge link group. In Section 4, we re-prove key lemmas and propositions with some modification, if necessary, to the original statements established in [1]. Finally, Section 5 is devoted to a new proof of the main theorem.

2. MAIN STATEMENT

For a rational number $r \in \hat{\mathbb{Q}} := \mathbb{Q} \cup \{\infty\}$, let $K(r)$ be the 2-bridge link of slope r , which is defined as the sum $(S^3, K(r)) = (B^3, t(\infty)) \cup (B^3, t(r))$ of rational tangles of slope ∞ and r . The common boundary $\partial(B^3, t(\infty)) = \partial(B^3, t(r))$ of the rational tangles is identified with the *Conway sphere* $(\mathbf{S}^2, \mathbf{P}) := (\mathbb{R}^2, \mathbb{Z}^2)/H$, where H is the group of isometries of the Euclidean plane \mathbb{R}^2 generated by the π -rotations around the points in the lattice \mathbb{Z}^2 . Let \mathbf{S} be the 4-punctured sphere $\mathbf{S}^2 - \mathbf{P}$ in the link complement $S^3 - K(r)$. Any essential simple loop in \mathbf{S} , up to isotopy, is obtained as the image of a line of slope $s \in \hat{\mathbb{Q}}$ in $\mathbb{R}^2 - \mathbb{Z}^2$ by the covering projection onto \mathbf{S} . The (unoriented) essential simple loop in \mathbf{S} so obtained is denoted by α_s . We also denote by α_s the conjugacy class of an element of $\pi_1(\mathbf{S})$ represented by (a suitably oriented) α_s . Then the *link group* $G(K(r)) := \pi_1(S^3 - K(r))$ is identified with $\pi_1(\mathbf{S})/\langle\langle \alpha_\infty, \alpha_r \rangle\rangle$, where $\langle\langle \cdot \rangle\rangle$ denotes the normal closure.

Let \mathcal{D} be the *Farey tessellation*, whose ideal vertex set is identified with $\hat{\mathbb{Q}}$. For each $r \in \hat{\mathbb{Q}}$, let Γ_r be the group of automorphisms of \mathcal{D} generated by reflections in the edges of \mathcal{D} with an endpoint r , and let $\hat{\Gamma}_r$ be the group

generated by Γ_r and Γ_∞ . Then the region, R , bounded by a pair of Farey edges with an endpoint ∞ and a pair of Farey edges with an endpoint r forms a fundamental domain of the action of $\hat{\Gamma}_r$ on \mathbb{H}^2 (see Figure 1). Let I_1 and I_2 be the closed intervals in $\hat{\mathbb{R}}$ obtained as the intersection with $\hat{\mathbb{R}}$ of the closure of R . Suppose that r is a rational number with $0 < r < 1$. (We may always assume this except when we treat the trivial knot and the trivial 2-component link.) Write $r = [m_1, m_2, \dots, m_k]$, where $k \geq 1$, $(m_1, \dots, m_k) \in (\mathbb{Z}_+)^k$, and $m_k \geq 2$. Then the above intervals are given by $I_1 = [0, r_1]$ and $I_2 = [r_2, 1]$, where

$$r_1 = \begin{cases} [m_1, m_2, \dots, m_{k-1}] & \text{if } k \text{ is odd,} \\ [m_1, m_2, \dots, m_{k-1}, m_k - 1] & \text{if } k \text{ is even,} \end{cases}$$

$$r_2 = \begin{cases} [m_1, m_2, \dots, m_{k-1}, m_k - 1] & \text{if } k \text{ is odd,} \\ [m_1, m_2, \dots, m_{k-1}] & \text{if } k \text{ is even.} \end{cases}$$

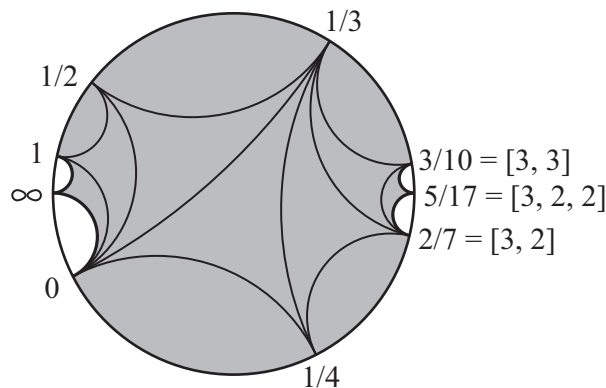


FIGURE 1. A fundamental domain of $\hat{\Gamma}_r$ in the Farey tessellation (the shaded domain) for $r = 5/17 = [3, 2, 2]$.

We recall the following fact ([3, Proposition 4.6 and Corollary 4.7] and [1, Lemma 7.1]) which describes the role of $\hat{\Gamma}_r$ in the study of 2-bridge link groups.

Proposition 2.1. (1) *If two elements s and s' of $\hat{\mathbb{Q}}$ belong to the same orbit $\hat{\Gamma}_r$ -orbit, then the unoriented loops α_s and $\alpha_{s'}$ are homotopic in $S^3 - K(r)$.*

(2) *For any $s \in \hat{\mathbb{Q}}$, there is a unique rational number $s_0 \in I_1 \cup I_2 \cup \{\infty, r\}$ such that s is contained in the $\hat{\Gamma}_r$ -orbit of s_0 . In particular, α_s is homotopic to α_{s_0} in $S^3 - K(r)$. Thus if $s_0 \in \{\infty, r\}$, then α_s is null-homotopic in $S^3 - K(r)$.*

The following theorem proved in [1] and to be re-proved in Section 5 of the present paper shows that the converse to Proposition 2.1(2) also holds.

Theorem 2.2. *The loop α_s is null-homotopic in $S^3 - K(r)$ if and only if s belongs to the $\hat{\Gamma}_r$ -orbit of ∞ or r . In other words, if $s \in I_1 \cup I_2$, then α_s is not null-homotopic in $S^3 - K(r)$.*

3. UPPER PRESENTATIONS OF 2-BRIDGE LINK GROUPS

Throughout this paper, the set $\{a, b\}$ denotes the standard meridian-generator of the rank 2 free group $\pi_1(B^3 - t(\infty))$, which is specified as in [1, Section 3]. For a positive rational number q/p , where p and q are relatively prime positive integers, let u_r be the word in $\{a, b\}$ obtained as follows. (For a geometric description, see [1, Remark 1].) Set $\epsilon_i = (-1)^{\lfloor iq/p \rfloor}$, where $\lfloor x \rfloor$ is the greatest integer not exceeding x .

(1) If p is odd, then

$$u_{q/p} = a \hat{u}_{q/p} b^{(-1)^q} \hat{u}_{q/p}^{-1},$$

$$\text{where } \hat{u}_{q/p} = b^{\epsilon_1} a^{\epsilon_2} \dots b^{\epsilon_{p-2}} a^{\epsilon_{p-1}}.$$

(2) If p is even, then

$$u_{q/p} = a \hat{u}_{q/p} a^{-1} \hat{u}_{q/p}^{-1},$$

$$\text{where } \hat{u}_{q/p} = b^{\epsilon_1} a^{\epsilon_2} \dots a^{\epsilon_{p-2}} b^{\epsilon_{p-1}}.$$

Then $u_r \in F(a, b) \cong \pi_1(B^3 - t(\infty))$ is represented by the simple loop α_r , and the link group $G(K(r))$ with $r > 0$ has the following presentation, called the *upper presentation*:

$$\begin{aligned} G(K(r)) &= \pi_1(S^3 - K(r)) \cong \pi_1(B^3 - t(\infty)) / \langle \langle \alpha_r \rangle \rangle \\ &\cong F(a, b) / \langle \langle u_r \rangle \rangle \cong \langle a, b \mid u_r \rangle. \end{aligned}$$

We recall the definition of the sequence $S(r)$ and the cyclic sequence $CS(r)$ of slope r defined in [1], both of which are read from the single relator u_r of the upper presentation of $G(K(r))$. We first fix some definitions and notation. Let X be a set. By a *word* in X , we mean a finite sequence $x_1^{\epsilon_1} x_2^{\epsilon_2} \dots x_n^{\epsilon_n}$ where $x_i \in X$ and $\epsilon_i = \pm 1$. Here we call $x_i^{\epsilon_i}$ the *i-th letter* of the word. For two words u, v in X , by $u \equiv v$ we denote the *visual equality* of u and v , meaning that if $u = x_1^{\epsilon_1} \dots x_n^{\epsilon_n}$ and $v = y_1^{\delta_1} \dots y_m^{\delta_m}$ ($x_i, y_j \in X$; $\epsilon_i, \delta_j = \pm 1$), then $n = m$ and $x_i = y_i$ and $\epsilon_i = \delta_i$ for each $i = 1, \dots, n$. The length of a word v is denoted by $|v|$. A word v in X is said to be *reduced* if v does not contain xx^{-1} or $x^{-1}x$ for any $x \in X$. A word is said to be *cyclically reduced* if all its cyclic permutations are reduced. A *cyclic word* is defined to be the set of all cyclic permutations of a cyclically reduced word. By (v) we denote the cyclic word associated with

a cyclically reduced word v . Also by $(u) \equiv (v)$ we mean the *visual equality* of two cyclic words (u) and (v) . In fact, $(u) \equiv (v)$ if and only if v is visually a cyclic shift of u .

Definition 3.1. (1) Let v be a nonempty reduced word in $\{a, b\}$. Decompose v into

$$v \equiv v_1 v_2 \cdots v_t,$$

where, for each $i = 1, \dots, t-1$, all letters in v_i have positive (resp., negative) exponents, and all letters in v_{i+1} have negative (resp., positive) exponents. Then the sequence of positive integers $S(v) := (|v_1|, |v_2|, \dots, |v_t|)$ is called the *S-sequence of v* .

(2) Let (v) be a nonempty cyclic word in $\{a, b\}$. Decompose (v) into

$$(v) \equiv (v_1 v_2 \cdots v_t),$$

where all letters in v_i have positive (resp., negative) exponents, and all letters in v_{i+1} have negative (resp., positive) exponents (taking subindices modulo t). Then the *cyclic sequence* of positive integers $CS(v) := ((|v_1|, |v_2|, \dots, |v_t|))$ is called the *cyclic S-sequence of (v)* . Here, the double parentheses denote that the sequence is considered modulo cyclic permutations.

(3) A nonempty reduced word v in $\{a, b\}$ is said to be *alternating* if $a^{\pm 1}$ and $b^{\pm 1}$ appear in v alternately, i.e., neither $a^{\pm 2}$ nor $b^{\pm 2}$ appears in v . A cyclic word (v) is said to be *alternating* if all cyclic permutations of v are alternating. In the latter case, we also say that v is *cyclically alternating*.

Definition 3.2. For a rational number r with $0 < r \leq 1$, let $G(K(r)) = \langle a, b | u_r \rangle$ be the upper presentation. Then the symbol $S(r)$ (resp., $CS(r)$) denotes the *S-sequence* $S(u_r)$ of u_r (resp., *cyclic S-sequence* $CS(u_r)$ of (u_r)), which is called the *S-sequence of slope r* (resp., the *cyclic S-sequence of slope r*).

The following is cited from [1]. Since its proof in [1] is irrelevant to the modification to be performed in the present paper, we adopt the proof as it is.

Lemma 3.3. [1, Proposition 4.2] *For the positive rational number $r = q/p$, the sequence $S(r)$ has length $2q$, and it represents the cyclic sequence $CS(r)$. Moreover the cyclic sequence $CS(r)$ is invariant by the half-rotation; that is, if $s_j(r)$ denotes the j -th term of $S(r)$ ($1 \leq j \leq 2q$), then $s_j(r) = s_{q+j}(r)$ for every integer j ($1 \leq j \leq q$).*

4. NEW PROOF FOR SMALL CANCELLATION CONDITIONS OF 2-BRIDGE LINK GROUPS

In this section, we give new proofs to several lemmas and propositions with some modification, if necessary, to the original statements established in [1, Section 4]. These will play crucial roles in the new proof of Theorem 2.2.

In the remainder of this paper unless specified otherwise, we suppose that r is a rational number with $0 < r \leq 1$, and write r as a continued fraction:

$$r = [m_1, m_2, \dots, m_k],$$

where $k \geq 1$, $(m_1, \dots, m_k) \in (\mathbb{Z}_+)^k$ and $m_k \geq 2$ unless $k = 1$.

Lemma 4.1. *For a rational number $r = [m_1, m_2, \dots, m_k]$ with $0 < r < 1$, let \tilde{r} be a rational number defined as*

$$\tilde{r} = \begin{cases} [m_1 - 1, m_2, m_3, \dots, m_k] & \text{if } m_1 \geq 2; \\ [m_2 + 1, m_3, m_4, \dots, m_k] & \text{if } m_1 = 1. \end{cases}$$

Then we have

$$r = \begin{cases} \tilde{r}/(1 + \tilde{r}) & \text{if } m_1 \geq 2; \\ 1 - \tilde{r} & \text{if } m_1 = 1. \end{cases}$$

Proof. If $m_1 \geq 2$, then letting $a := 1/\tilde{r} = m_1 - 1 + [m_2, \dots, m_k]$ we have

$$r = [m_1, m_2, \dots, m_k] = 1/(1 + a) = 1/(1 + 1/\tilde{r}) = \tilde{r}/(1 + \tilde{r}),$$

as required.

On the other hand, if $m_1 = 1$, then letting $b := 1/\tilde{r} - 1 = m_2 + [m_3, \dots, m_k]$ we have

$$r = [m_1, m_2, \dots, m_k] = 1/(1 + 1/b) = 1/(1 + \tilde{r}/(1 - \tilde{r})) = 1 - \tilde{r},$$

as required. □

Proposition 4.2. *For a rational number $r = [m_1, m_2, \dots, m_k]$ with $0 < r < 1$, let \tilde{r} be a rational number defined as in Lemma 4.1. Put $CS(\tilde{r}) = ((a_1, a_2, \dots, a_t, a_1, a_2, \dots, a_t))$. Then the following hold.*

(1) *If $m_1 \geq 2$, then*

$$CS(r) = ((a_1 + 1, a_2 + 1, \dots, a_t + 1, a_1 + 1, a_2 + 1, \dots, a_t + 1)).$$

(2) *If $m_1 = 1$, then every a_i is at least 2, and either*

$$CS(r) = ((2, b_1 \langle 1 \rangle, 2, b_2 \langle 1 \rangle, \dots, 2, b_t \langle 1 \rangle, 2, b_1 \langle 1 \rangle, 2, b_2 \langle 1 \rangle, \dots, 2, b_t \langle 1 \rangle))$$

or

$$CS(r) = ((2, b_t\langle 1 \rangle, \dots, 2, b_2\langle 1 \rangle, 2, b_1\langle 1 \rangle, 2, b_t\langle 1 \rangle, \dots, 2, b_2\langle 1 \rangle, 2, b_1\langle 1 \rangle)),$$

where $b_i = a_i - 2$ for every i , and the symbol " $b_i\langle 1 \rangle$ " represents b_i successive 1's. (Here if $b_i = 0$ for some i , then $b_i\langle 1 \rangle$ represents the empty subsequence.)

Proof. (1) Let $m_1 \geq 2$. Write $\tilde{r} = q/p$, where p and q are relatively prime positive integers. By Lemma 4.1, $r = \tilde{r}/(1 + \tilde{r}) = q/(p + q)$. It then follows from Lemma 3.3 that both the sequences $S(r)$ and $S(\tilde{r})$, and hence both the cyclic sequences $CS(r)$ and $CS(\tilde{r})$, have the same length $2q$. Recall from [1, Lemma 4.8] that if $s_j(r)$ denotes the j -th term of the sequence $S(r)$, then $s_j(r) = \lfloor j(1/r) \rfloor_* - \lfloor (j-1)(1/r) \rfloor_*$, where $\lfloor x \rfloor_*$ is the greatest integer smaller than x . Since $r = \tilde{r}/(1 + \tilde{r}) = 1/(1/\tilde{r} + 1)$, we have

$$\begin{aligned} s_j(r) &= \lfloor j(1/r) \rfloor_* - \lfloor (j-1)(1/r) \rfloor_* \\ &= \lfloor j(1/\tilde{r} + 1) \rfloor_* - \lfloor (j-1)(1/\tilde{r} + 1) \rfloor_* \\ &= (\lfloor j(1/\tilde{r}) \rfloor_* + j) - (\lfloor (j-1)(1/\tilde{r}) \rfloor_* + (j-1)) \\ &= 1 + \lfloor j(1/\tilde{r}) \rfloor_* - \lfloor (j-1)(1/\tilde{r}) \rfloor_* \\ &= 1 + s_j(\tilde{r}), \end{aligned}$$

where $s_j(\tilde{r})$ denotes the j -th term of the sequence $S(\tilde{r})$, and hence the assertion follows.

(2) Let $m_1 = 1$. Then $\tilde{r} = [m_2 + 1, m_3, \dots, m_k]$ and $r = 1 - \tilde{r}$ by Lemma 4.1. Since $m_2 + 1 \geq 2$, (1) implies that every term of $CS(\tilde{r})$ is at least 2, that is, every a_i is at least 2.

To prove the remaining assertion, let f_1 be the reflection of $(B^3, t(\infty))$ in a "horizontal" disk bounded by α_0 , and let f_2 be the half Dehn twist of $(B^3, t(\infty))$ along the "vertical" disk bounded by α_∞ . Then the automorphisms $(f_i)_*$ of $\pi_1(B^3 - t(\infty)) = F(a, b)$ induced by f_i are given by

$$(f_1)_*(a, b) = (a, b) \quad (f_2)_*(a, b) = (a, b^{-1})$$

Let f be the composition $f_2 f_1$. Then by the above observation, we have $f_*(a, b) = (a, b^{-1})$. On the other hand, f maps α_r to $f_2(f_1(\alpha_r)) = f_2(\alpha_{-r}) = \alpha_{1-r} = \alpha_{\tilde{r}}$. Thus f_* sends the cyclic word (u_r) to the cyclic word $(u_{\tilde{r}})$ or $(u_{\tilde{r}}^{-1})$. Since $f_*^2 = 1$, this implies that f_* sends the cyclic word $(u_{\tilde{r}})$ to the cyclic word (u_r) or (u_r^{-1}) . Thus the cyclic word (u_r) or (u_r^{-1}) is obtained from $(u_{\tilde{r}})$ by replacing b with b^{-1} . In this process, a subword, w , of $(u_{\tilde{r}})$ with $S(w) = (1, a_i, 1)$, say, $w = b^{-1}(abab \cdots ab)a^{-1}$ or $b^{-1}(abab \cdots a)b^{-1}$ according to whether a_i is even or odd, is transformed to a subword $w' = b(ab^{-1}ab^{-1} \cdots ab^{-1})a^{-1}$ or

$b(ab^{-1}ab^{-1}\cdots a)b$, respectively, of $(u_r^{\pm 1})$ with $S(w') = (2, (a_i - 2)\langle 1 \rangle, 2)$. Since the cyclic sequence $CS(u_r^{-1})$ is the reverse of the cyclic sequence $CS(u_r) = CS(r)$, the assertion now follows. \square

Throughout the remainder of this paper, we assume the following well-ordering \preceq .

Definition 4.3. Let \mathfrak{A} be the set of all rational numbers greater than 0 and less than or equal to 1. We define a well-ordering \preceq on \mathfrak{A} by $r_1 \preceq r_2$ if and only if one of the following conditions holds, where $r_1 = [l_1, l_2, \dots, l_h]$ and $r_2 = [n_1, n_2, \dots, n_t]$.

- (i) $h < t$.
- (ii) $h = t$ and there is a positive integer $j \leq h = t$ such that $l_i = n_i$ for every $i < j$ and $l_j \leq n_j$.

It should be noted that a rational number \tilde{r} defined in Lemma 4.1 is a predecessor of $r = [m_1, m_2, \dots, m_k]$ with respect to \preceq .

Now we are able to give a new proof to the following lemma whose statement is a part of [1, Proposition 4.3]. Note that the remaining part of [1, Proposition 4.3] is not necessary in the present paper.

Lemma 4.4. *For a rational number $r = [m_1, m_2, \dots, m_k]$, we have the following.*

- (1) *Suppose $k = 1$, i.e., $r = 1/m_1$. Then $CS(r) = ((m_1, m_1))$.*
- (2) *Suppose $k \geq 2$. Then $CS(r)$ consists of m_1 and $m_1 + 1$.*

Proof. We prove (1) and (2) together by transfinite induction with respect to the well-ordering \preceq defined in Definition 4.3. The base step is the case when $r = [1]$. In this case, $u_r = ab^{-1}$, and so $CS(r) = ((1, 1))$, as desired. To prove the inductive step, we consider two cases separately.

Case 1. $m_1 \geq 2$.

In this case, put $\tilde{r} = [m_1 - 1, m_2, \dots, m_k]$ as in Lemma 4.1. Then clearly $\tilde{r} \prec r$. By the inductive hypothesis, $CS(\tilde{r}) = ((m_1 - 1, m_1 - 1))$ provided $k = 1$, and $CS(\tilde{r})$ consists of $m_1 - 1$ and m_1 provided $k \geq 2$. So by Proposition 4.2(1), $CS(r) = ((m_1, m_1))$ provided $k = 1$, and $CS(r)$ consists of m_1 and $m_1 + 1$ provided $k \geq 2$, as desired.

Case 2. $m_1 = 1$.

In this case, it immediately follows from Proposition 4.2(2) that $CS(r)$ consists of $1 = m_1$ and $2 = m_1 + 1$, as desired. \square

We also give a new proof to the following proposition whose statement is precisely the same as [1, Proposition 4.5].

Proposition 4.5. *For $r = [m_1, m_2, \dots, m_k]$, the cyclic sequence $CS(r)$ has a decomposition $((S_1, S_2, S_1, S_2))$ which satisfies the following.*

- (1) *Each S_i is symmetric, that is, the sequence obtained from S_i by reversing the order is equal to S_i . (Here, S_1 is empty if $k = 1$.)*
- (2) *Each S_i occurs only twice on the cyclic sequence $CS(r)$.*
- (3) *The subsequence S_1 begins and ends with $m_1 + 1$.*
- (4) *The subsequence S_2 begins and ends with m_1 .*

Proof. The proof proceeds by transfinite induction with respect to the well-ordering \preceq defined in Definition 4.3. We take the case when $r = [m_1]$ as the base step. In this case, $CS(r) = ((m_1, m_1))$ by Lemma 4.4(1). Putting $S_1 = \emptyset$ and $S_2 = (m_1)$, the assertion clearly holds. To prove the inductive step, we consider two cases separately.

Case 1. $m_1 \geq 2$ and $k \geq 2$.

Put $\tilde{r} = [m_1 - 1, m_2, \dots, m_k]$ as in Lemma 4.1. Then clearly $\tilde{r} \prec r$. By the inductive hypothesis, $CS(\tilde{r}) = ((\tilde{S}_1, \tilde{S}_2, \tilde{S}_1, \tilde{S}_2))$, where \tilde{S}_1 and \tilde{S}_2 are symmetric subsequences of $CS(\tilde{r})$ such that each \tilde{S}_i occurs only twice in $CS(\tilde{r})$, \tilde{S}_1 begins and ends with m_1 (provided that \tilde{S}_1 is nonempty), and such that \tilde{S}_2 begins and ends with $m_1 - 1$. Write

$$\tilde{S}_1 = (a_1, \dots, a_{t_1}) \quad \text{and} \quad \tilde{S}_2 = (a_{t_1+1}, \dots, a_{t_2}),$$

and then take

$$S_1 = (a_1 + 1, \dots, a_{t_1} + 1) \quad \text{and} \quad S_2 = (a_{t_1+1} + 1, \dots, a_{t_2} + 1).$$

Clearly S_1 begins and ends with $m_1 + 1$, and S_2 begins and ends with m_1 . Also since \tilde{S}_1 and \tilde{S}_2 are symmetric by the inductive hypothesis, S_1 and S_2 are also symmetric. Moreover, by Proposition 4.2(1), we have $CS(r) = ((S_1, S_2, S_1, S_2))$. It remains to show that each S_i occurs only twice in $CS(r)$. If S_1 occurred more than twice in $((S_1, S_2, S_1, S_2))$, \tilde{S}_1 also would occur more than twice in $((\tilde{S}_1, \tilde{S}_2, \tilde{S}_1, \tilde{S}_2))$, a contradiction. Similarly, S_2 also occurs only twice in $CS(r)$.

Case 2. $m_1 = 1$ and $k \geq 2$.

Put $\tilde{r} = [m_2 + 1, m_3, \dots, m_k]$ as in Lemma 4.1. Then clearly $\tilde{r} \prec r$. By the inductive hypothesis, $CS(\tilde{r}) = ((\tilde{S}_1, \tilde{S}_2, \tilde{S}_1, \tilde{S}_2))$, where \tilde{S}_1 and \tilde{S}_2 are symmetric subsequences of $CS(\tilde{r})$ such that each \tilde{S}_i occurs only twice in $CS(\tilde{r})$, \tilde{S}_1 begins

and ends with $m_2 + 2$ (provided that \tilde{S}_1 is nonempty), and such that \tilde{S}_2 begins and ends with $m_2 + 1$. If $k = 2$, then $r = [1, m_2]$ with $m_2 \geq 2$ and $\tilde{r} = [m_2 + 1]$; so $CS(\tilde{r}) = ((m_2 + 1, m_2 + 1))$ by Lemma 4.4(1). Then take

$$S_1 = (2) \quad \text{and} \quad S_2 = ((m_2 - 1)\langle 1 \rangle).$$

On the other hand, if $k \geq 3$, then write

$$\tilde{S}_1 = (a_1, \dots, a_{t_1}) \quad \text{and} \quad \tilde{S}_2 = (a_{t_1+1}, \dots, a_{t_2}).$$

Here $a_1 = a_{t_1} = m_2 + 2 \geq 3$ and $a_{t_1+1} = a_{t_2} = m_2 + 1 \geq 2$. Now take

$$S_1 = (2, b_{t_1+1}\langle 1 \rangle, 2, \dots, 2, b_{t_2}\langle 1 \rangle, 2) \quad \text{and} \quad S_2 = (b_1\langle 1 \rangle, 2, \dots, 2, b_{t_1}\langle 1 \rangle),$$

where $b_i = a_i - 2$ for every i . In either case, we see that S_1 begins and ends with $2 = m_1 + 1$, S_2 begins and ends with $1 = m_1$, and that S_1 and S_2 are symmetric because \tilde{S}_1 and \tilde{S}_2 are symmetric by the inductive hypothesis. Moreover, Proposition 4.2(2) implies that either $CS(r) = ((S_1, S_2, S_1, S_2))$ or $CS(r) = ((\overleftarrow{\tilde{S}_1}, \overleftarrow{\tilde{S}_2}, \overleftarrow{\tilde{S}_1}, \overleftarrow{\tilde{S}_2}))$, where the symbol “ $\overleftarrow{S_i}$ ” denotes the reverse of S_i . But since S_1 and S_2 are symmetric, we actually have $CS(r) = ((S_1, S_2, S_1, S_2))$ in either case. It remains to show that each S_i occurs only twice in $CS(r)$. If S_1 occurred more than twice in $((S_1, S_2, S_1, S_2))$, \tilde{S}_2 also would occur more than twice in $((\tilde{S}_1, \tilde{S}_2, \tilde{S}_1, \tilde{S}_2))$, a contradiction. For the assertion for S_2 , note that S_2 begins and ends with m_2 successive 1's, and that the maximum number of consecutive occurrences of 1 in $((S_1, S_2, S_1, S_2))$ is m_2 . So if S_2 occurred more than twice in $((S_1, S_2, S_1, S_2))$, \tilde{S}_1 also would occur more than twice in $((\tilde{S}_1, \tilde{S}_2, \tilde{S}_1, \tilde{S}_2))$, a contradiction. \square

In order to prove Theorem 2.2, we keep the idea of applying small cancellation theory as in [1, Sections 5 and 6]. Briefly speaking, we adopt [1, Section 5] as it is to show that the upper presentation $G(K(r)) = \langle a, b \mid u_r \rangle$ with $0 < r < 1$ satisfies the small cancellation conditions $C(4)$ and $T(4)$. And then we investigate properties of van Kampen's diagrams over the presentation $G(K(r)) = \langle a, b \mid u_r \rangle$ with boundary label being cyclically alternating as in [1, Section 6]. Sections 5 and 6 in [1] are indeed irrelevant to the modification that we are performing in the present paper. Due to van Kampen's Lemma which is a classical result in combinatorial group theory (see [2]), we obtain the fact that if a cyclically alternating word w equals the identity in $G(K(r))$, then its cyclic word (w) contains a subword z of $(u_r^{\pm 1})$ such that the S -sequence of z is (S_1, S_2, ℓ) or (ℓ, S_2, S_1) for some positive integer ℓ , where $CS(r) = ((S_1, S_2, S_1, S_2))$ is as in Proposition 4.5. In particular, we obtain the following.

Corollary 4.6. [1, Corollary 6.4] *Let $r = [m_1, m_2, \dots, m_k]$ with $0 < r < 1$. For a rational number s with $0 < s \leq 1$, if α_s is null-homotopic in $S^3 - K(r)$, then the following hold.*

- (1) *If $k = 1$, namely $r = [m_1]$, then $CS(s)$ contains a term bigger than or equal to m_1 .*
- (2) *If $k \geq 2$, then $CS(s)$ contains (S_1, S_2) or (S_2, S_1) as a subsequence, where $CS(r) = ((S_1, S_2, S_1, S_2))$ is as in Proposition 4.5.*

Remark 4.7. In [1, Corollary 6.4], it is mistakenly stated that if α_s is null-homotopic in $S^3 - K(r)$, then $CS(s)$ contains (S_1, S_2) or (S_2, S_1) as a subsequence, regardless of $k \geq 1$. It should be noted that if $k = 1$ and every term of $CS(s)$ is bigger than m_1 , then $CS(s)$ does not contain (S_1, S_2) or (S_2, S_1) as a subsequence, because, in this case, S_1 is empty and $S_2 = (m_1)$, i.e., $(S_1, S_2) = (m_1) = (S_2, S_1)$.

5. NEW PROOF OF THEOREM 2.2

In this section, we prove the only if part of Theorem 2.2, that is, we prove that for any $s \in I_1 \cup I_2$, α_s is not null-homotopic in $S^3 - K(r)$. The if part is [3, Corollary 4.7].

The following lemma which plays an important role in the proof of Theorem 2.2 has the same statement as [1, Lemma 7.3], but is re-proved by transfinite induction.

Lemma 5.1. *Let $r = [m_1, m_2, \dots, m_k]$ with $0 < r < 1$, and let $CS(r) = ((S_1, S_2, S_1, S_2))$ be as in Proposition 4.5. Suppose that a rational number s with $0 < s \leq 1$ has a continued fraction expansion $s = [l_1, \dots, l_t]$, where $t \geq 1$, $(l_1, \dots, l_t) \in (\mathbb{Z}_+)^t$, and $l_t \geq 2$ unless $t = 1$. Suppose also that $CS(s)$ satisfies the following condition:*

- (i) *If $k = 1$, namely $r = [m_1]$, then $CS(s)$ contains a term bigger than or equal to m_1 .*
- (ii) *If $k \geq 2$, then $CS(s)$ contains (S_1, S_2) or (S_2, S_1) as a subsequence.*

Then the following hold.

- (1) $t \geq k$.
- (2) $l_i = m_i$ for each $i = 1, \dots, k - 1$.
- (3) Either $l_k \geq m_k$ or both $l_k = m_k - 1$ and $t > k$.

Proof. The proof proceeds by transfinite induction with respect to the well-ordering \preceq defined in Definition 4.3. We take the case when $r = [m_1]$ as the base. By hypothesis (i), $CS(s)$ contains a term bigger than or equal to m_1 .

Then Lemma 4.4 implies that either $l_1 \geq m_1$ or both $l_1 = m_1 - 1$ and $t \geq 2$, so that the assertion clearly holds. Now we prove the inductive step. Let \tilde{r} be defined as in Lemma 4.1. Then clearly $\tilde{r} \prec r$.

Case 1. $m_1 \geq 2$ and $k \geq 2$.

In this case, $\tilde{r} = [m_1 - 1, m_2, \dots, m_k]$. By Proposition 4.5, S_1 begins and ends with $m_1 + 1$, and S_2 begins and ends with m_1 . Hence if $CS(s)$ contains (S_1, S_2) or (S_2, S_1) as a subsequence, then $CS(s)$ contains both a term m_1 and a term $m_1 + 1$. By Lemma 4.4, the only possibility is that $l_1 = m_1$ and $t \geq 2$. Now let $\tilde{s} = [l_1 - 1, \dots, l_t]$. Then we see from Proposition 4.2(1) that $CS(\tilde{s})$ contains $(\tilde{S}_1, \tilde{S}_2)$ or $(\tilde{S}_2, \tilde{S}_1)$ as a subsequence, where $CS(\tilde{r}) = ((\tilde{S}_1, \tilde{S}_2, \tilde{S}_1, \tilde{S}_2))$. By the inductive hypothesis, we have $t \geq k$, $l_i = m_i$ for each $i = 1, \dots, k - 1$, and either $l_k \geq m_k$ or both $l_k = m_k - 1$ and $t > k$, which proves the assertion.

Case 2. $m_1 = 1$ and $k \geq 2$.

In this case, $\tilde{r} = [m_2 + 1, m_3, \dots, m_k]$. Arguing as in Case 1, $CS(s)$ contains both a term $m_1 = 1$ and a term $m_1 + 1 = 2$. By Lemma 4.4, the only possibility is that $l_1 = m_1 = 1$ and $t \geq 2$. Now let $\tilde{s} = [l_2 + 1, \dots, l_t]$. Then we see from Proposition 4.2(2) that $CS(\tilde{s})$ contains a term greater than or equal to $m_2 + 1$ provided $k = 2$ and that $CS(\tilde{s})$ contains $(\tilde{S}_1, \tilde{S}_2)$ or $(\tilde{S}_2, \tilde{S}_1)$ as a subsequence provided $k \geq 3$, where $CS(\tilde{r}) = ((\tilde{S}_1, \tilde{S}_2, \tilde{S}_1, \tilde{S}_2))$. By the inductive hypothesis, we have $t \geq k$, $l_i = m_i$ for each $i = 2, \dots, k - 1$, and either $l_k \geq m_k$ or both $l_k = m_k - 1$ and $t > k$. This together with $l_1 = m_1$ proves the assertion. \square

Remark 5.2. We can easily see that the a rational number s with $0 < s \leq 1$ satisfies the conclusion of Lemma 5.1 if and only if s lies in the open interval $(r_1, r_2) = (0, 1] - (I_1 \cup I_2)$, where r_1 and r_2 are rational numbers such that $I_1 = [0, r_1]$ and $I_2 = [r_2, 1]$, introduced in Section 2.

We are now in a position to prove the only if part of Theorem 2.2.

Proof of the only if part of Theorem 2.2. Since the exceptional cases $r = \infty$ and $r = 1$ can be treated in the same way as in [1, Section 7], we assume $0 < r < 1$. Consider a 2-bridge link $K(r)$, and pick a rational number s from $I_1 \cup I_2$. Suppose on the contrary that α_s is null-homotopic in $S^3 - K(r)$, namely $u_s = 1$ in $G(K(r))$. If $0 < s \leq 1$, then by Corollary 4.6, $CS(s)$ contains a term greater than or equal to m_1 provided $r = [m_1]$ or otherwise $CS(s)$ contains (S_1, S_2) or (S_2, S_1) as a subsequence, where $CS(r) = ((S_1, S_2, S_1, S_2))$ as in Proposition 4.5. But then by Lemma 5.1 together with Remark 5.2, we have $s \notin I_1 \cup I_2$, a contradiction. So the only possibility is $s = 0$. Then, as mentioned at the end of Section 4 (also see [1, Theorem 6.3]), u_s must contain a subword

z of $(u_r^{\pm 1})$ such that the S -sequence of z is (S_1, S_2, ℓ) or (ℓ, S_2, S_1) for some positive integer ℓ . Note that the length of such a subword z is strictly greater than p , half the length of $(u_r^{\pm 1})$, where $r = q/p$. Since $0 < r < 1$, we have $p \geq 2$. So, the word $u_0 = ab$ cannot contain such a subword, a contradiction. This completes the proof of the only if part of Theorem 2.2. \square

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